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Best Monotone Approximation in $L_\infty[0, 1]$

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1. INTRODUCTION

If $f(x)$ is a Lebesgue measurable function on $[0, 1]$ and $p > 1$, let $f_p(x)$ be the unique best L_p-approximant to $f(x)$ by non-decreasing functions on $[0, 1]$. If

$$\lim_{p \to \infty} f_p(x) \equiv f_\infty(x)$$

exists a.e., then $f_\infty(x)$ is a best L_∞-approximant to $f(x)$ by non-decreasing functions. In this case, we say that the Polya algorithm converges and $f_\infty(x)$ is *a best best L_∞-approximant.*

In [1], it is shown that if $f(x)$ is quasi-continuous, then the Polya algorithm converges. A function $f(x)$ is *quasi-continuous* if

$$\begin{aligned} &\lim_{y \to x^+} f(y) \text{ exists for all } 0 \leqslant x < 1, \\ &\lim_{y \to x^-} f(y) \text{ exists for all } 0 < x \leqslant 1. \end{aligned} \tag{1.1}$$

In [2], it is shown that if $f(x)$ is only assumed to be Lebesgue measurable, then the algorithm may fail to converge. In this paper, we show that the condition that $f(x)$ be quasi-continuous can be relaxed to the condition that $f(x)$ can be uniformly approximated by simple Lebesgue measurable functions where the one-sided limits shown in (1.1) need only exist at a few select points. Besides extending the result of [1], we believe the construction of $f_\infty(x)$ as given in this paper gives a clearer picture of what $f_\infty(x)$ is, even when $f(x)$ is continuous.

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2. The Construction of $f_\infty(x)$

Let $f(x) = \sum_{i=1}^{N} a_i X_{E_i}(x)$ be a Lebesgue measurable simple function. For convenience we assume $a_i - a_j \neq a_k - a_m$ for all $(i, j) \neq (k, m)$. We construct partition points of $[0, 1]$ according to the following steps.

Step 1. Let

$$b_{11} = \operatorname*{ess\,max}_{x<y} (f(x) - f(y))^+.$$

If $b_{11} = 0$, then f is essentially non-decreasing on $[0, 1]$ with essential jump discontinuities at $\{z_{111}, \ldots, z_{11k}\}$.

If $b_{11} > 0$, then let

$$x_{11} = \inf\{x\colon \exists y_0 > x \ni f(x) - f(y_0) = b_{11} \text{ and } m(t < x | f(t) - f(y_0) = b_{11}) > 0\}$$

and

$$y_{11} = \sup\{y > x_{11} : \exists x_0 < y \ni f(x_0) - f(y) = b_{11} \text{ and } m(t > y | f(x_0) - f(t) = b_{11}) > 0\}.$$

In the preceding definitions, $m(S)$ denotes the Lebesgue measure of S.

Step 2.1. If $x_{11} = 0$, go to step 2.2. If $x_{11} > 0$, let

$$b_{21} = \operatorname*{ess\,max}_{x<y<x_{11}} (f(x) - f(y))^+.$$

If $b_{21} = 0$, then f is essentially non-decreasing on $[0, x_{11}]$, with essential jump discontinuities at $\{z_{211}, \ldots, z_{21k}\}$.

If $b_{21} > 0$, then let

$$x_{21} = \inf\{x < x_{11} : \exists y_0 > x \ni f(x) - f(y_0) = b_{21} \text{ and } m(t < x | f(t) - f(y_0) = b_{21} > 0\}$$

and

$$y_{21} = \sup\{y < x_{11} : \exists x_0 < x_{11}, x_0 < y \ni f(x_0) - f(y) = b_{21} \text{ and } m(y < t < x_{11} | f(x_0) - f(t) = b_{21}) > 0\}.$$

Step 2.2. If $x_{11} = 0$ and $y_{11} = 1$, stop.
If $x_{11} > 0$ and $y_{11} = 1$, go to the next step.

If $x_{11} > 0$ and $y_{11} < 1$, let

$$b_{22} = \operatorname*{ess\,max}_{y_{11}<x<y} (f(x) - f(y))^+.$$

If $b_{22} = 0$, then f is essentially non-decreasing on $[y_{11}, 1]$, with essential jump discontinuities at $\{z_{221}, \ldots, z_{22k}\}$.

If $b_{22} = 0$ and $b_{21} = 0$, step.

If $b_{22} = 0$ and $b_{21} > 0$, go to the next step.

If $b_{22} > 0$, then let

$$x_{22} = \inf\{x > y_{11} : \exists y_0 > x \ni f(x) - f(y_0) = b_{22} \text{ and } m(y_{11} < t < x \mid f(t) - f(y_0) = b_{22}) > 0\}$$

and

$$y_{22} = \sup\{y > x_{22} : \exists x_0 > y_{11}, x_0 < y \ni f(x) - f(y) = b_{22} \text{ and } m(t > y \mid f(x_0) - f(t) = b_{22}) > 0\}.$$

Continue to define the x_{ij}, y_{ij}, and z_{ijk} in this manner over the remaining intervals $[0, x_{21}]$, $[y_{21}, x_{11}]$, $[y_{11}, x_{22}]$ and $[y_{22}, 1]$. Since f is a simple function, this process terminates after finitely many steps.

Let $P = \{x_{ij}\} \cup \{y_{ij}\} \cup \{z_{ijk}\} \cup \{0, 1\}$ and then let $\{t_1, \ldots, t_n\}$ be a relabeling of P in increasing order.

We now define $f_\infty(x)$, which is a best L_∞ approximation to $f(x)$ by non-decreasing functions.

Step 1. If $b_{11} = 0$, then f is essentially non-decreasing on $[0, 1]$. By the definition of $\{z_{11k}\}$, if t_i, $t_{i+1} \in P$, then f is essentially constant on (t_i, t_{i+1}). Let B_{11i} be that constant. Then for all $x \in (t_i, t_{i+1}]$, define $f_\infty(x) = B_{11i}$ and we are finished.

If $b_{11} > 0$, then $\exists x_{11}^1 > x_{11}$ and $y_{11}^1 < y_{11}$ such that $f(x_{11}^1) - f(y_{11}^1) = b_{11}$. Then for $x \in [x_{11}, y_{11}]$ define

$$f_\infty(x) = \tfrac{1}{2}[f(x_{11}^1) + f(y_{11}^1)] \equiv A_{11}.$$

Step 2.1. If $b_{21} = 0$, then f is essentially non-decreasing on $[0, x_{11}]$. By the definition of $\{z_{21k}\}$, if $(t_i, t_{i+1}] \subseteq [0, x_{11}]$ then f is essentially constant on $(t_i, t_{i+1}]$. Let that constant be B_{21i}. For $x \in (t_i, t_{i+1}]$, define

$$f_\infty(x) = \min\{B_{21i}, A_{11}\}.$$

If $b_{21} > 0$, then $\exists x_{21}^1$, y_{21}^1 such that $x_{21} \leqslant x_{21}^1 < y_{21}^1 \leqslant y_{21}$ and $f(x_{21}^1) - f(y_{21}^1) = b_{21}$. Then for all $x \in [x_{21}, y_{21})$, define

$$f_\infty(x) = \min\{\tfrac{1}{2}[f(x_{21}^1) + f(y_{21}^1)], A_{11}\} \equiv A_{21}.$$

Note that if $A_{21} = A_{11}$, this forces $f_\infty(x) = A_{11}$ for all $x \in (y_{21}, x_{11}]$.

Step 2.2. If $b_{22} = 0$, then f is essentially non-decreasing on $[y_{11}, 1]$. By the definiton of $\{z_{22k}\}$, if $(t_i, t_{i+1}] \subseteq (y_{11}, 1]$, then f is essentially constant on $(t_i, t_{i+1}]$. Let that constant be B_{22i}, and for $x \in (t_i, t_{i+1}]$ define

$$f_\infty(x) = \max\{B_{22i}, A_{11}\}.$$

If $b_{22} > 0$ (and the interval $(x_{22}, y_{22}]$ is defined), then $\exists x_{22}^1$, y_{22}^1 such that $x_{22} \leqslant x_{22}^1 < y_{22}^1 \leqslant y_{22}$ and $f(x_{22}^1) - f(y_{22}^1) = b_{22}$.
Then for all $x \in [x_{22}, y_{22}]$ define

$$f_\infty(x) = \max\{\tfrac{1}{2}[f(x_{22}^1) + f(y_{22}^1)], A_{11}\} \equiv A_{22}.$$

Step 3.1. If b_{31} is defined and $b_{31} = 0$, then f is essentially non-decreasing on $[0, x_3]$ and $f_\infty(x)$ is defined as in Step 2.1.
If $b_{31} > 0$, then x_{31} and y_{31} are defined, and $f_\infty(x)$ is defined on $(x_{31}, y_{31}]$ as in Step 2.1.

Step 3.2. If b_{32} is defined and $b_{32} = 0$, then f is essentially non-decreasing on $[y_{21}, x_{11}]$. By the definition of $\{z_{32k}\}$, if $(t_i, t_{i+1}] \subseteq [y_{21}, x_{11}]$ then f is essentially constant on $(t_i, t_{i+1}]$. Call that constant B_{32i} and for all $x \in (t_i, t_{i+1}]$ define

$$f_\infty(x) = \min[\max\{A_{21}, B_{32i}\}, A_{11}].$$

If $b_{32} > 0$, then $\exists x_{32}^1$, y_{32}^1 such that $x_{32} \leqslant x_{32}^1 < y_{32}^1 \leqslant y_{32}$ and $f(x_{32}^1) - f(y_{32}^1) = b_{32}$.
Then for all $x \in (x_{32}, y_{32}]$ define

$$f_\infty(x) = \min[\max\{A_{21}, \tfrac{1}{2}[f(x_{32}^1) + f(y_{32}^1)]\}, A_{11}].$$

The definition of $f_\infty(x)$ for all subsequent steps follows the patterns established above.

THEOREM 3.1. *If $f(x)$ is a simple Lebesgue measurable function and if $\lim_{x\to x_{ij}^+} f(x)$ and $\lim_{x\to y_{ij}^-} f(x)$ exist for each x_{ij}, y_{ij} as described in Section 2 then $f_p(x)$ can be chosen so that*

$$\lim_{p\to\infty} f_p(x) = f_\infty(x)$$

uniformly on $[0, 1]$.

Proof. For each interval $[x_{ij}, y_{ij}]$, define

$$x_{ij}^{11} = \sup\{x_{ij} \leqslant x \leqslant y_{ij} : m[x_{ij} < t < x \mid f(t) \neq f(x_{ij}^1)] = 0\}$$

and

$$y_{ij}^{11} = \inf\{x_{ij} \leqslant y \leqslant y_{ij} : m[y < t < y_{ij} \mid f(t) \neq f(y_{ij}^1)] = 0\}.$$

Because f is a simple function and since $\lim_{x \to x_{ij}^+} f(x)$ and $\lim_{x \to y_{ij}^-} f(x)$ exist, then by the definition of x_{ij} and y_{ij} we have $x_{ij}^{11} > x_{ij}$ and $y_{ij}^{11} < y_{ij}$.

Now let $\varepsilon > 0$ and suppose $b_{11} > 0$. Choose p_{11} so that if $p \geqslant p_{11}$, then

$$\begin{aligned} (\tfrac{1}{2}b_{11} + \varepsilon)^p (x_{11}^{11} - x_{11}) &> (\tfrac{1}{2}b_{11})^p, \\ (\tfrac{1}{2}b_{11} + \varepsilon)^p (y_{11} - y_{11}^{11}) &> (\tfrac{1}{2}b_{11})^p. \end{aligned} \tag{2.1}$$

We have that $f_\infty(x)$ is constant on $(x_{11}, y_{11}]$. If $f_p(x) \geqslant f_\infty(x) + \varepsilon$ for some $x \in (x_{11}, y_{11}^{11}]$, then $f_p(x) \geqslant f_\infty(x) + \varepsilon$ for all $x \in (y_{11}^{11}, y_{11}]$ since $f_p(x)$ is a non-decreasing function. It would follow that

$$|f(x) - f_p(x)| \geqslant \tfrac{1}{2}b_{11} + \varepsilon$$

for almost all $x \in (y_{11}^{11}, y_{11}]$. Also,

$$|f(x) - f_\infty(x)| \leqslant \tfrac{1}{2}b_{11} \qquad \text{for almost all } x \in [0, 1].$$

Hence

$$\int_{x_{11}}^{y_{11}} |f(x) - f_p(x)|^p \, dm \geqslant (\tfrac{1}{2}b_{11} + \varepsilon)^p (y_{11} - y_{11}^{11})$$

and

$$\int_0^1 |f(x) - f_\infty(x)|^p \, dm \leqslant (\tfrac{1}{2}b_{11})^p. \tag{3.3}$$

Hence if $p_{11} \leqslant p$, inequalities (3.2) and (3.3) together with (3.1) imply that $f_\infty(x)$ is a better Lp-approximant to $f(x)$ than $f_p(x)$, a contradiction.

Hence $f_p(x) < f_\infty(x) + \varepsilon$ for all $x \in (x_{11}, y_{11}^{11}]$. A similar argument shows that $f_p(x) < f_\infty(x) + \varepsilon$ for all $x \in (y_{11}^{11}, y_{11}]$ as well. It can also be established using similar arguments that $f_p(x) > f_\infty(x) - \varepsilon$ for $p \geqslant p_{11}$ and all $x \in (x_{11}, y_{11}]$.

In the same way, for each interval $(x_{ij}, y_{ij}]$ there is a p_{ij} so that $|f_p(x) - f_\infty(x)| < \varepsilon$ for $p \geqslant p_{ij}$ and for all $x \in (x_{ij}, y_{ij}]$. Letting $p_0 = \max\{p_{ij}\}$, we have

$$|f_p(x) - f_\infty(x)| < \varepsilon$$

for all $p \geqslant p_0$ and all $x \in \bigcup (x_{ij}, y_{ij}]$.

Finally, from the construction of the points $\{z_{ijk}\}$, it is clear that

$$|f_p(x) - f_\infty(x)| < \varepsilon \qquad \text{for } p \geqslant p_0 \text{ and all } x \in [0, 1] \backslash \bigcup (x_{ij}, y_{ij}]$$

and the theorem is proved.

THEOREM 3.2. *If f is the uniform limit of a sequence $\{f_n\}$ of simple functions satisfying the hypothesis of Theorem* 3.1, *then* $\lim_{p \to \infty} f_p(x)$ *exists uniformly on* $[0, 1]$.

Proof. Let $\varepsilon > 0$. Choose n so large that for all $x \in [0, 1]$, we have

$$f(x) - \varepsilon/6 < f_n(x) < f(x) + \varepsilon/6.$$

By the monotony property of L_p-approximation (see [3]), we have

$$f_p(x) - \varepsilon/3 < f_{np}(x) < f_p(x) + \varepsilon/3 \tag{3.4}$$

for all $p > 1$ and all x. It follows that $|f_p(x) - f_{np}(x)| < \varepsilon/3$ for all $p > 1$ and all $x \in [0, 1]$. By Theorem 3.1, we can choose $p_0 > 0$ so large that

$$|f_{np}(x) - f_{nq}(x)| < \varepsilon/3 \quad \text{for } p, q \geqslant p_0 \text{ and all } x \in [0, 1].$$

It follows that $|f_p(x) - f_q(x)| < \varepsilon$ for all p, $q \geqslant p_0$ and all $x \in [0, 1]$. Hence $\lim_{p \to \infty} f_p(x)$ exists uniformly.

If we now let $p \to \infty$ in (3, 4), we obtain $|f_{n\infty}(x) - f_\infty(x)| \leqslant \varepsilon/3$ for sufficiently large n. Hence

$$\lim_{n \to \infty} f_{n\infty}(x) = f_\infty(x)$$

for $x \in [0, 1]$ uniformly.

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